

Symplectic and Hamiltonian Deformations of Gabor Frames

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May 7, 2013

Abstract

We study symplectic deformations of Gabor frames, using the covariance properties of the Heisenberg operators. This allows us to recover in a very simple way known results. We thereafter propose a general deformation scheme by Hamiltonian isotopies, which are paths of Hamiltonian flows. We define and study in detail a weak notion of Hamiltonian deformations, using ideas from semiclassical analysis due to Heller and Hagedorn. This method can be easily implemented using symplectic integrators.

1 Introduction

The theory of Gabor frames (or Weyl–Heisenberg frames as they are also called) is a rich and expanding topic of harmonic analysis. It has many applications in time-frequency analysis, signal theory, and mathematical physics. The aim of this article is to initiate a systematic study of the symplectic transformation properties of Gabor frames, both in the linear and nonlinear cases. Strangely enough, the use of symplectic techniques in the theory of Gabor frames is very often ignored; one example (among many others) being Casazza’s seminal paper [9] on modern tools for Weyl–Heisenberg frame theory, where the word “symplectic” does not appear a single time in the 127 pages of this paper! There are however exceptions: in Gröchenig’s treatise [27] the metaplectic representation is used to study various symmetries; the same applies to the recent paper by Pfander *et al.* [45], elaborating on earlier work [29] by Han and Wang, where symplectic transformations are exploited to study various properties of Gabor frames.

In this paper we consider the notion of deformation of a Gabor system using the *Hamiltonian isotopies* we introduce in section 5. A Hamiltonian

isotopy is a curve $(f_t)_{0 \leq t \leq 1}$ of diffeomorphisms of phase space \mathbb{R}^{2n} starting at the identity, and such that there exists a Hamiltonian function H , usually time-dependent such that the (generalized) phase flow $(f_t^H)_t$ determined by the Hamilton equations

$$\dot{x} = \partial_p H(x, p, t) \quad , \quad \dot{p} = -\partial_x H(x, p, t) \quad (1)$$

consists of precisely the mappings f_t for $0 \leq t \leq 1$. It follows that a Hamiltonian isotopy consists of symplectomorphisms (or canonical transformations, as they are called in Physics). Given a Gabor system $\mathcal{G}(\phi, \Lambda)$ with window (or atom) ϕ and lattice Λ we want to find a working definition of the deformation of $\mathcal{G}(\phi, \Lambda)$ by a Hamiltonian isotopy $(f_t)_{0 \leq t \leq 1}$. While it is clear that the deformed lattice should be the image $\Lambda_t = f_t(\Lambda)$ of the original lattice Λ , it is less clear what the deformation $\phi_t = f_t(\phi)$ of the window ϕ should be. A clue is given by the linear case: assume that the mappings f_t are linear, *i.e.* symplectic matrices S_t ; assume in addition that there exists an infinitesimal symplectic transformation X such that $S_t = e^{tX}$ for $0 \leq t \leq 1$. Then $(S_t)_t$ is the flow determined by the Hamiltonian function

$$H(x, p) = -\frac{1}{2}(x, p)^T J X(x, p) \quad (2)$$

where J is the standard symplectic matrix. There exists a one-parameter group of unitary operators $(\hat{S}_t)_t$ satisfying the operator Schrödinger equation

$$i\hbar \frac{d}{dt} \hat{S}_t = H(x, -i\hbar \partial_x) \hat{S}_t$$

where the formally self-adjoint operator $H(x, -i\hbar \partial_x)$ is obtained by replacing formally p with $-i\hbar \partial_x$ in (2); the matrices S_t and the operators \hat{S}_t correspond to each other via the metaplectic representation of the symplectic group. This suggests that we define the deformation of the initial window ϕ by $\phi_t = \hat{S}_t \phi$. It turns out that this definition is satisfactory, because it allows to recover, setting $t = 1$, known results on the image of Gabor frames by linear symplectic transformations. This example is thus a good guideline; however one encounters difficulties as soon as one want to extend it to more general situations. While it is “reasonably” easy to see what one should do when the Hamiltonian isotopy consists of an arbitrary path of symplectic matrices (this will be done in section 4), it is not clear at all what a “good” definition should be in the general nonlinear case: this is discussed in section 6, where we suggest that a natural choice would be to extend the linear case by requiring that ϕ_t should be the solution of the Schrödinger equation

$$i\hbar \frac{d}{dt} \phi_t = \hat{H} \phi_t$$

associated with the Hamiltonian function H determined by the equality $(f_t)_{0 \leq t \leq 1} = (f_t^H)_{0 \leq t \leq 1}$; the Hamiltonian operator \hat{H} would then be associated with the function H by using, for instance, the Weyl correspondence. Since the method seems to be difficult to study theoretically and to implement numerically, we propose what we call a notion of *weak deformation*, where the exact definition of the transformation $\phi \mapsto \phi_t$ of the window ϕ is replaced with a correspondence used in semiclassical mechanics, and which consists in propagating the “center” of a sufficiently sharply peaked initial window ϕ (for instance a coherent state, or a more general Gaussian) along the Hamiltonian trajectory. This definition coincides with the definition already given in the linear case, and has the advantage of being easily computable using the method of symplectic integrators (which we review in section 5.3) since all what is needed is the knowledge of the phase flow determined by a certain Hamiltonian function. Finally we discuss possible extensions of our method.

Notation and terminology

The generic point of the phase space $\mathbb{R}^{2n} \equiv \mathbb{R}^n \times \mathbb{R}^n$ is denoted by $z = (x, p)$ where we have set $x = (x_1, \dots, x_n)$, $p = (p_1, \dots, p_n)$. The scalar product of two vectors, say p and x , is denoted by $p \cdot x$ or simply px . When matrix calculations are performed, z, x, p are viewed as column vectors. We will equip \mathbb{R}^{2n} with the standard symplectic structure

$$\sigma(z, z') = p \cdot x' - p' \cdot x;$$

in matrix notation $\sigma(z, z') = (z')^T J z$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (0 and I are the $n \times n$ zero and identity matrices). The symplectic group of \mathbb{R}^{2n} is denoted by $\text{Sp}(n)$; it consists of all linear automorphisms of \mathbb{R}^{2n} such that $\sigma(Sz, Sz') = \sigma(z, z')$ for all $z, z' \in \mathbb{R}^{2n}$. Working in the canonical basis $\text{Sp}(n)$ is identified with the group of all real $2n \times 2n$ matrices S such that $S^T J S = J$ (or, equivalently, $S J S^T = J$).

We will write $dz = dx dp$ where $dx = dx_1 \cdots dx_n$ and $dp = dp_1 \cdots dp_n$. The scalar product on $L^2(\mathbb{R}^n)$ is defined by

$$(\psi|\phi) = \int_{\mathbb{R}^n} \psi(x) \overline{\phi(x)} dx$$

and the associated norm is denoted by $\|\cdot\|$. The Schwartz space of rapidly decreasing functions is denoted by $\mathcal{S}(\mathbb{R}^n)$ and its dual (the space of tempered distributions) by $\mathcal{S}'(\mathbb{R}^n)$.

2 Gabor Frames

Gabor frames are a generalization of the usual notion of basis; see for instance Gröchenig [27], Feichtinger and Gröchenig [16], Balan *et al.* [3], Heil [32], Casazza [9] for a detailed treatment of this topic. In what follows we give a slightly modified version of the usual definition, better adapted to the study of symplectic symmetries.

2.1 Definition

Let ϕ be a non-zero square integrable function (hereafter called *window*) on \mathbb{R}^n , and a lattice Λ in \mathbb{R}^{2n} , *i.e.* a discrete subset of \mathbb{R}^{2n} . The associated \hbar -Gabor system is the set of square-integrable functions

$$\mathcal{G}(\phi, \Lambda) = \{\hat{T}^{\hbar}(z)\phi : z \in \Lambda\}$$

where $\hat{T}^{\hbar}(z) = e^{-i\sigma(\hat{z}, z)/\hbar}$ is the Heisenberg operator. The action of this operator is explicitly given by the formula

$$\hat{T}^{\hbar}(z_0)\phi(x) = e^{i(p_0x - p_0x_0/2)/\hbar}\phi(x - x_0) \quad (3)$$

(see *e.g.* [22, 24, 39]; it will be justified in section 4.2). We will call the Gabor system $\mathcal{G}(g, \Lambda)$ a \hbar -frame for $L^2(\mathbb{R}^n)$, if there exist constants $a, b > 0$ (the *frame bounds*) such that

$$a\|\psi\|^2 \leq \sum_{z_0 \in \Lambda} |(\psi|\hat{T}^{\hbar}(z_0)\phi)|^2 \leq b\|\psi\|^2 \quad (4)$$

for every square integrable function ψ on \mathbb{R}^n . When $a = b$ the \hbar -frame $\mathcal{G}(g, \Lambda)$ is said to be *tight*.

Remark 1 *The product $(\psi|\hat{T}^{\hbar}(z_0)\phi)$ is, up to the factor $(2\pi\hbar)^{-n}$, Woodward's cross-ambiguity function [50]; it is up to a (symplectic Fourier transform) the cross-Wigner distribution $W(\psi, \phi)$ as was already observed by Klauder [37]; see [19, 22, 24].*

2.2 Rescaling properties

For the choice $\hbar = 1/2\pi$ the notion of \hbar -Gabor frame coincides with the usual notion of Gabor frame as found in the literature. In fact, in this case, writing $\hat{T}(z) = \hat{T}^{1/2\pi}(z)$ and $p = \omega$, we have

$$|(\psi|\hat{T}(z)\phi)| = |(\psi|\tau(z)\phi)|$$

where $\tau(z)$ is the modulation operator defined by

$$\tau(z_0)\phi(x) = e^{2\pi i\omega_0 x}\phi(x - x_0)$$

for $z_0 = (x_0, \omega_0)$. The two following elementary results can be used to go from one definition to the other:

Proposition 2 *Let $D^h = \begin{pmatrix} I & 0 \\ 0 & 2\pi\hbar I \end{pmatrix}$. The system $\mathcal{G}(\phi, \Lambda)$ is a Gabor frame if and only if $\mathcal{G}(\phi, D^h\Lambda)$ is a \hbar -Gabor frame.*

Proof. We have $\widehat{T}^h(x_0, 2\pi\hbar p_0) = \widehat{T}(x_0, p_0)$ where $\widehat{T}(x_0, p_0) = \widehat{T}^{1/2\pi}(x_0, p_0)$. By definition $\mathcal{G}(\phi, \Lambda)$ is a Gabor frame if and only if

$$a\|\psi\|^2 \leq \sum_{z_0 \in \Lambda} |(\psi|\widehat{T}(z_0)\phi)|^2 \leq b\|\psi\|^2$$

for every $\psi \in L^2(\mathbb{R}^n)$ that is

$$a\|\psi\|^2 \leq \sum_{(x_0, p_0) \in \Lambda} |(\psi|\widehat{T}(x_0, p_0)\phi)|^2 \leq b\|\psi\|^2;$$

this inequality is equivalent to

$$a\|\psi\|^2 \leq \sum_{(x_0, p_0) \in \Lambda} |(\psi|\widehat{T}^h(x_0, 2\pi\hbar p_0)\phi)|^2 \leq b\|\psi\|^2$$

that is to

$$a\|\psi\|^2 \leq \sum_{(x_0, (2\pi\hbar)^{-1}p_0) \in \Lambda} |(\psi|\widehat{T}^h(x_0, p_0)\phi)|^2 \leq b\|\psi\|^2$$

hence the result since $(x_0, (2\pi\hbar)^{-1}p_0) \in \Lambda$ means that $(x_0, p_0) \in D^h\Lambda$. ■

We can also rescale simultaneously the lattice and the window (“change of Planck’s constant”):

Proposition 3 *Let $\mathcal{G}(\phi, \Lambda)$ be a Gabor system, and set*

$$\phi^h(x) = (2\pi\hbar)^{-n/2}\phi(x/\sqrt{2\pi\hbar}). \quad (5)$$

Then $\mathcal{G}(\phi, \Lambda)$ is a frame if and only if $\mathcal{G}(\phi^h, \sqrt{2\pi\hbar}\Lambda)$ is a \hbar -frame.

Proof. We have $\phi^h = \widehat{M}_{1/\sqrt{2\pi\hbar}I,0}\phi$ where $\widehat{M}_{1/\sqrt{2\pi\hbar}I,0} \in \text{Mp}(n)$ has projection

$$M_{1/\sqrt{2\pi\hbar}} = \begin{pmatrix} (2\pi\hbar)^{1/2}I & 0 \\ 0 & (2\pi\hbar)^{-1/2}I \end{pmatrix}$$

on $\text{Sp}(n)$ (see Appendix A). The Gabor system $\mathcal{G}(\phi^h, \sqrt{2\pi\hbar}\Lambda)$ is a \hbar -frame if and only

$$a\|\psi\|^2 \leq \sum_{z_0 \in \sqrt{2\pi\hbar}\Lambda} |(\psi|\widehat{T}(z_0)\widehat{M}_{1/\sqrt{2\pi\hbar}I,0}\phi)|^2 \leq b\|\psi\|^2$$

for every $\psi \in L^2(\mathbb{R}^n)$, that is, taking the symplectic covariance formula (6) into account, if and only if

$$a\|\psi\|^2 \leq \sum_{z_0 \in \sqrt{2\pi\hbar}\Lambda} |(\widehat{M}_{\sqrt{2\pi\hbar}I,0}\psi|\widehat{T}((2\pi\hbar)^{-1/2}x_0, (2\pi\hbar)^{1/2}p_0)\phi)|^2 \leq b\|\psi\|^2.$$

But this inequality is equivalent to

$$a\|\psi\|^2 \leq \sum_{z_0 \in D^h\Lambda} |(\psi|\widehat{T}(z_0)\phi)|^2 \leq b\|\psi\|^2$$

and one concludes using Proposition 2. ■

Remark 4 In Appendix A we state a rescaling property for the covering projection $\pi^h : \widehat{S} \mapsto S$ of metaplectic group $\text{Mp}(n)$ onto $\text{Sp}(n)$ (formula (59)).

3 Symplectic Covariance

The following formula will play a fundamental role in our study of symplectic covariance properties of frames. It relates Heisenberg–Weyl operators, linear symplectic transformations, and metaplectic operators (we refer to Appendix A for a concise review of the metaplectic group $\text{Mp}(n)$ and its properties).

Let $\widehat{S} \in \text{Mp}(n)$ have projection $\pi^h(\widehat{S}) = S \in \text{Sp}(n)$. Then

$$\widehat{T}^h(z)\widehat{S} = \widehat{S}\widehat{T}^h(S^{-1}z). \quad (6)$$

For a proof see *e.g.* [22, 24, 39]; one easy way is to prove this formula separately for each generator \widehat{J} , $\widehat{M}_{L,m}$, \widehat{V}_P of the metaplectic group.

3.1 A first covariance result

Gabor frames behave well under symplectic transformations of the lattice (or, equivalently, under metaplectic transformations of the window). Let $\widehat{S} \in \text{Mp}(n)$ have projection $S \in \text{Sp}(n)$. The following result is well-known, and appears in many places in the literature (see *e.g.* Gröchenig [27], Pfander *et al.* [45]). Our proof is somewhat simpler since it exploits the symplectic covariance property of the Heisenberg–Weyl operators.

Proposition 5 *Let $\phi \in L^2(\mathbb{R}^n)$ (or $\phi \in \mathcal{S}(\mathbb{R}^n)$). A Gabor system $\mathcal{G}(\phi, \Lambda)$ is a \hbar -frame if and only if $\mathcal{G}(\widehat{S}\phi, S\Lambda)$ is a \hbar -frame; when this is the case both frames have the same bounds. In particular, $\mathcal{G}(\phi, \Lambda)$ is a tight \hbar -frame if and only if $\mathcal{G}(\phi, \Lambda)$ is.*

Proof. Using the symplectic covariance formula (6) we have

$$\begin{aligned} \sum_{z \in S\Lambda} |(\psi|\widehat{T}^h(z)\widehat{S}\phi)|^2 &= \sum_{z \in S\Lambda} |(\psi|\widehat{S}\widehat{T}^h(S^{-1}z)\phi)|^2 \\ &= \sum_{z \in \Lambda} |(\widehat{S}^{-1}\psi|\widehat{T}^h(z)\phi)|^2 \end{aligned}$$

and hence, since $\mathcal{G}(\phi, \Lambda)$ is a \hbar -frame,

$$a\|\widehat{S}^{-1}\psi\|^2 \leq \sum_{z \in S\Lambda} |(\psi|\widehat{T}^h(z)\widehat{S}\phi)|^2 \leq b\|\widehat{S}^{-1}\psi\|^2.$$

The result follows since $\|\widehat{S}^{-1}\psi\| = \|\psi\|$ because metaplectic operators are unitary; the case $\phi \in \mathcal{S}(\mathbb{R}^n)$ is similar since metaplectic operators are linear automorphisms of $\mathcal{S}(\mathbb{R}^n)$. ■

Remark 6 *The result above still holds when one assumes that the window ϕ belongs to the Feichtinger algebra $S_0(\mathbb{R}^n)$ (see Appendix B and the discussion at the end of the paper).*

3.2 Gaussian frames

The problem of constructing Gabor frames $\mathcal{G}(\phi, \Lambda)$ in $L^2(\mathbb{R})$ with an arbitrary window ϕ and lattice Λ is difficult and has been tackled by many authors (see for instance the comments in [28], also [45]). Very little is known about the existence of frames in the general case. We however have the following characterization of Gaussian frames which extends a classical result of Lyubarskii [41] and Seip and Wallstén [47]:

Proposition 7 Let $\phi_0^h(x) = (\pi\hbar)^{-n/4} e^{-|x|^2/2\hbar}$ (the standard centered Gaussian) and $\Lambda_{\alpha\beta} = \alpha\mathbb{Z}^n \times \beta\mathbb{Z}^n$ with $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. Then $\mathcal{G}(\phi_0^h, \Lambda_{\alpha\beta})$ is a frame if and only if $\alpha_j\beta_j < 2\pi\hbar$ for $1 \leq j \leq n$.

Proof. Bourouhiya [5] proves this for $\hbar = 1/2\pi$; the result for arbitrary $\hbar > 0$ follows using Proposition 3. ■

It turns out that using the result above one can construct infinitely many symplectic Gaussian frames using the theory of metaplectic operators:

Proposition 8 Let ϕ_0^h be the standard Gaussian. The Gabor system $\mathcal{G}(\phi_0^h, \Lambda_{\alpha\beta})$ is a frame if and only if $\mathcal{G}(\widehat{S}\phi_0^h, S\Lambda_{\alpha\beta})$ is a frame (with same bounds) for every $\widehat{S} \in \text{Mp}(n)$. Writing S in block-matrix form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the window $\widehat{S}\phi_0^h$ is the Gaussian

$$\widehat{S}\phi_0^h(x) = \left(\frac{1}{\pi\hbar}\right)^{n/4} (\det X)^{1/4} e^{-\frac{1}{2\hbar}(X+iY)x \cdot x} \quad (7)$$

where

$$X = -(CA^T + DB^T)(AA^T + BB^T)^{-1} \quad (8)$$

$$Y = (AA^T + BB^T)^{-1} \quad (9)$$

are symmetric matrices, and $X > 0$.

Proof. That $\mathcal{G}(\phi_0^h, \Lambda_{\alpha\beta})$ is a frame if and only if $\mathcal{G}(\widehat{S}\phi_0^h, S\Lambda_{\alpha\beta})$ is a frame follows from Proposition 5. To calculate $\widehat{S}\phi_0^h$ it suffices to apply formulas (64) and (65) in Appendix A. ■

3.3 An example

Let us choose $\hbar = 1/2\pi$ and consider the rotations

$$S_t = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad (10)$$

(we assume $n = 1$). The matrices (S_t) form a one-parameter subgroup of the symplectic group $\text{Sp}(1)$. To (S_t) corresponds a unique one-parameter subgroup (\widehat{S}_t) of the metaplectic group $\text{Mp}(1)$ such that $S_t = \pi^{1/2\pi}(\widehat{S}_t)$. It follows from formula (A1) in Appendix A that $\widehat{S}_t\phi$ is given for $t \neq k\pi$ (k integer) by the explicit formula

$$\widehat{S}_t\phi_0^h(x) = i^{m(t)} \left(\frac{1}{2\pi i |\sin t|} \right)^{1/2} \int_{-\infty}^{\infty} e^{2\pi i W(x, x', t)} \phi_0^h(x') dx'$$

where $m(t)$ is an integer (the “Maslov index”) and

$$W(x, x', t) = \frac{1}{2 \sin t} ((x^2 + x'^2) \cos t - 2xx').$$

Remark 9 *The metaplectic operators \widehat{S}_t are the “fractional Fourier transforms” familiar from time-frequency analysis (see e.g. Almeida [1], Namias [43]).*

Applying Proposition 5 we recover without any calculation the results of Kaiser [35] (Theorem 1 and Corollary 2) about rotations of Gabor frames; in our notation:

Corollary 10 *Let $\mathcal{G}(\phi, \Lambda)$ be a frame; then $\mathcal{G}(\widehat{S}\phi, S\Lambda)$ is a frame for every $\widehat{S} \in \text{Mp}(n)$.*

4 Symplectic Deformations of Gabor Frames

The symplectic covariance property of Gabor frames studied above can be interpreted as a first result on Hamiltonian deformations of frames because, as we will see, every symplectic matrix is the value of the flow (at some time t) of a Hamiltonian function which is a homogeneous quadratic polynomial (with time-dependent coefficients) in the coordinates x_j, p_k . We will in fact extend this result to deformations by affine flows corresponding to the case where the Hamiltonian is an arbitrary quadratic function of these coordinates.

4.1 The linear case

The first example in subsection 3.3 (the fractional Fourier transform) can be interpreted as a statement about continuous deformations of Gabor frames. For instance, assume that $S_t = e^{tX}$, X in the Lie algebra $\mathfrak{sp}(n)$ of the symplectic group $\text{Sp}(n)$ (it is the algebra of all $2n \times 2n$ matrices X such that $XJ + JX^T = 0$; when $n = 1$ it reduces to the condition $\text{Tr } X = 0$; see [19, 22]). It is then easy to check that S_t can be identified with the flow determined by the Hamilton equations $\dot{z} = J\partial_z H$ for the function

$$H(z) = -\frac{1}{2}z^T(JX)z \tag{11}$$

that is

$$\frac{d}{dt}S_t = XS_t. \tag{12}$$

A fundamental observation is now that to the path of symplectic matrices $t \mapsto S_t$, $0 \leq t \leq 1$ corresponds a unique path $t \mapsto \hat{S}_t$, $0 \leq t \leq 1$, of metaplectic operators such that $\hat{S}_0 = I$ and $\hat{S}_1 = \hat{S}$ (see Appendix A). This path satisfies the operator Schrödinger equation

$$i\hbar \frac{d}{dt} \hat{S}_t = \hat{H} \hat{S}_t \quad (13)$$

where \hat{H} is the Weyl quantization of the function H . Collecting these facts, one sees that $\mathcal{G}(\hat{S}\phi_0^{\hbar}, S\Lambda_{\alpha\beta})$ is obtained from the initial Gabor frame $\mathcal{G}(\phi_0^{\hbar}, \Lambda_{\alpha\beta})$ by a smooth deformation

$$t \mapsto \mathcal{G}(\hat{S}_t\phi_0^{\hbar}, S_t\Lambda_{\alpha\beta}) \quad , \quad 0 \leq t \leq 1. \quad (14)$$

More generally, let S be an arbitrary element of the symplectic group $\text{Sp}(n)$. The latter is connected so there exists a C^1 path (in fact infinitely many) $t \mapsto S_t$, $0 \leq t \leq 1$, joining the identity to S in $\text{Sp}(n)$. An essential result, generalizing the observations above, is the following:

Proposition 11 *Let $t \mapsto S_t$, $0 \leq t \leq 1$, be a path in $\text{Sp}(n)$ such that $S_0 = I$ and $S_1 = S$. There exists a Hamiltonian function $H = H(z, t)$ such that S_t is the phase flow determined by the Hamilton equations $\dot{z} = J\partial_z H$. Writing*

$$S_t = \begin{pmatrix} A_t & B_t \\ C_t & D_t \end{pmatrix} \quad (15)$$

the Hamiltonian function is the quadratic form

$$H(z, t) = \frac{1}{2}(\dot{D}_t C_t^T - \dot{C}_t D_t^T)x^2 + (\dot{D}_t A_t^T - \dot{C}_t B_t^T)p \cdot x + \frac{1}{2}(\dot{B}_t A_t^T - \dot{A}_t B_t^T)p^2 \quad (16)$$

where $\dot{A}_t = dA_t/dt$, etc.

Proof. The matrices S_t being symplectic we have $S_t^T J S_t = J$. Differentiating both side of this equality with respect to t and setting we get $\dot{S}_t^T J S_t + S_t^T J \dot{S}_t = 0$ or, equivalently,

$$J \dot{S}_t S_t^{-1} = -(S_t^T)^{-1} \dot{S}_t^T J = (J \dot{S}_t S_t^{-1})^T.$$

This equality can be rewritten $J \dot{S}_t S_t^{-1} = (J \dot{S}_t S_t^{-1})^T$ hence the matrix $J \dot{S}_t S_t^{-1}$ is symmetric. Set $J \dot{S}_t S_t^{-1} = M_t = M_t^T$; then

$$\dot{S}_t = X_t S_t \quad , \quad X_t = -J M_t \quad (17)$$

(it reduces to formula (12) when M_t is time-independent). Define now

$$H(z, t) = -\frac{1}{2}z^T(JX_t)z;$$

using (17) one verifies that the phase flow determined by H consists precisely of the symplectic matrices S_t and that H is given by formula (16). ■

Remark 12 *Formula (16) also follows from the more general formula (31) about Hamiltonian isotopies in Proposition (15) below.*

Exactly as above, to the path of symplectic matrices $t \mapsto S_t$ corresponds a path $t \mapsto \hat{S}_t$ of metaplectic operators such that $\hat{S}_0 = I$ and $\hat{S}_1 = \hat{S}$ satisfying the Schrödinger equation (13). Thus, it makes sense to consider smooth deformations (14) for arbitrary symplectic paths. This situation will be generalized to the nonlinear case in a moment.

4.2 Translations of Gabor systems

A particular simple example of transformation is that of the translations $T(z_0) : z \mapsto z + z_0$ in \mathbb{R}^{2n} . On the operator level they correspond to the Heisenberg–Weyl operators $\hat{T}^h(z_0)$. This correspondence is very easy to understand in terms of “quantization”: for fixed z_0 consider the Hamiltonian function

$$H(z) = \sigma(z, z_0) = p \cdot x_0 - p_0 \cdot x.$$

The corresponding Hamilton equations are just $\dot{x} = x_0$, $\dot{p} = p_0$ whose solutions are $x(t) = x(0) + tx_0$ and $p(t) = p(0) + tp_0$, that is $z(t) = T(tz_0)z(0)$. Let now

$$\hat{H} = \sigma(\hat{z}, z_0) = (-i\hbar\partial_x) \cdot x_0 - p_0 \cdot x.$$

be the “quantization” of H , and consider the Schrödinger equation

$$i\hbar\partial_t\phi = \sigma(\hat{z}, z_0)\phi.$$

Its solution is given by

$$\phi(x, t) = e^{-t\sigma(\hat{z}, z_0)/\hbar}\phi(x, 0) = \hat{T}^h(tz_0)\phi(x, 0)$$

(the second equality can be verified by a direct calculation, or using the Campbell–Hausdorff formula [19, 22, 24, 39]).

Translations act in a particularly simple way on Gabor frames; we write $T(z_1)\Lambda = \Lambda + z_1$.

Proposition 13 *Let $z_0, z_1 \in \mathbb{R}^{2n}$. A Gabor system $\mathcal{G}(\phi, \Lambda)$ is a \hbar -frame if and only if $\mathcal{G}(\widehat{T}^{\hbar}(z_0)\phi, T(z_1)\Lambda)$ is a \hbar -frame; the frame bounds are in this case the same for all values of z_0, z_1 .*

Proof. We will need the following well-known [19, 22, 24, 39] properties of the Heisenberg–Weyl operators:

$$\widehat{T}^{\hbar}(z)\widehat{T}^{\hbar}(z') = e^{i\sigma(z, z')/\hbar}\widehat{T}^{\hbar}(z')\widehat{T}^{\hbar}(z) \quad (18)$$

$$\widehat{T}^{\hbar}(z + z') = e^{-i\sigma(z, z')/2\hbar}\widehat{T}^{\hbar}(z)\widehat{T}^{\hbar}(z'). \quad (19)$$

Assume first $z_1 = 0$ and let us prove that $\mathcal{G}(\widehat{T}^{\hbar}(z_0)\phi, \Lambda)$ is a \hbar -frame if and only $\mathcal{G}(\phi, \Lambda)$ is. We have, using formula (18) and the unitarity of $\widehat{T}^{\hbar}(z_0)$,

$$\begin{aligned} \sum_{z \in \Lambda} |(\psi|\widehat{T}^{\hbar}(z)\widehat{T}^{\hbar}(z_0)\phi)|^2 &= \sum_{z \in \Lambda} |(\psi|e^{i\sigma(z, z_0)/\hbar}\widehat{T}^{\hbar}(z_0)\widehat{T}^{\hbar}(z)\phi)| \\ &= \sum_{z \in \Lambda} |(\psi|\widehat{T}^{\hbar}(z_0)\widehat{T}^{\hbar}(z)\phi)| \\ &= \sum_{z \in \Lambda} |(\widehat{T}^{\hbar}(-z_0)\psi|\widehat{T}^{\hbar}(z)\phi)|; \end{aligned}$$

it follows that the inequality

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\widehat{T}^{\hbar}(z)\widehat{T}^{\hbar}(z_0)\phi)|^2 \leq b\|\psi\|^2$$

is equivalent to

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\widehat{T}^{\hbar}(z)\phi)|^2 \leq b\|\psi\|^2$$

hence our claim in the case $z_1 = 0$. We next assume that $z_0 = 0$; we have, using this time formula (19),

$$\begin{aligned} \sum_{z \in T(z_1)\Lambda} |(\psi|\widehat{T}^{\hbar}(z)\phi)|^2 &= \sum_{z \in \Lambda} |(\psi|\widehat{T}^{\hbar}(z + z_1)\phi)|^2 \\ &= \sum_{z \in \Lambda} |(\psi|\widehat{T}^{\hbar}(z_1)\widehat{T}^{\hbar}(z)\phi)|^2 \\ &= \sum_{z \in \Lambda} |(\widehat{T}^{\hbar}(-z_1)\psi|\widehat{T}^{\hbar}(z)\phi)|^2 \end{aligned}$$

and one concludes as in the case $z_1 = 0$. The case of arbitrary z_0, z_1 immediately follows. ■

Identifying the group of translations with \mathbb{R}^{2n} the inhomogeneous (or affine) symplectic group $\text{ISp}(n)$ [8, 19] is the semi-direct product $\text{Sp}(n) \ltimes \mathbb{R}^{2n}$; the group law is given by

$$(S, z)(S', z') = (SS', z + Sz').$$

Using the conjugation relation

$$S^{-1}T(z_0)S = T(S^{-1}z_0) \quad (20)$$

one checks that $\text{ISp}(n)$ is isomorphic to the group of all affine transformations of \mathbb{R}^{2n} of the type $ST(z_0)$ (or $T(z_0)S$) where $S \in \text{Sp}(n)$.

The group $\text{ISp}(n)$ appears in a natural way when one considers Hamiltonians of the type

$$H(z, t) = \frac{1}{2}M(t)z \cdot z + m(t) \cdot z \quad (21)$$

where $M(t)$ is symmetric and $m(t)$ is a vector. In fact, the phase flow determined by the Hamilton equations for (21) consists of elements of $\text{ISp}(n)$. Assume for instance that the coefficients M and m are time-independent; the solution of Hamilton's equations $\dot{z} = JMz + Jm$ is

$$z_t = e^{tJM}z_0 + (JM)^{-1}(e^{tJM} - I)JM \quad (22)$$

provided that $\det M \neq 0$. When $\det M = 0$ the solution (22) is still formally valid and depends on the nilpotency degree of $X = JM$. Since $X = JM \in \mathfrak{sp}(n)$ we have $S_t = e^{tX} \in \text{Sp}(n)$; setting $\xi_t = X^{-1}(e^{tX} - I)u$ the flow (f_t^H) is thus given by

$$f_t^H = T(\xi_t)S_t \in \text{ISp}(n).$$

The metaplectic group $\text{Mp}(n)$ is a unitary representation of the double cover $\text{Sp}_2(n)$ of $\text{Sp}(n)$ (see Appendix A). There is an analogue when $\text{Sp}(n)$ is replaced with $\text{ISp}(n)$; it is the Weyl-metaplectic group $\text{WMP}(n)$, which consists of all products $\hat{T}(z_0)\hat{S}$; notice that formula (6), which we can rewrite

$$\hat{S}^{-1}\hat{T}^h(z)\hat{S} = \hat{T}^h(S^{-1}z). \quad (23)$$

is the operator version of formula (20) above.

5 The Group $\text{Ham}(n)$

In this section we review the basics of the modern theory of Hamiltonian mechanics from the symplectic point of view; for details we refer to [12, 34, 46]; we have also given elementary accounts in [22, 24].

5.1 Hamiltonian flows: properties

A Hamiltonian system (1) can be written in compact form as

$$\dot{z} = J\partial_z H(z, t) \quad (24)$$

where J is the standard symplectic matrix. The Hamiltonian function H is assumed to be twice continuously differentiable in z , and continuous in t . We denote by f_t^H the mapping $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ which to an initial condition z_0 associates the value $z = f_t^H(z_0)$ of the solution to (24) at time t . The family $(f_t^H)_t$ of all these mappings is called the phase flow determined by the Hamiltonian system (24).

It is often useful to replace the notion of flow as defined as above by that of *time-dependent flow* $(f_{t,t'}^H)$: $f_{t,t'}^H$ is the function such that $f_{t,t'}^H(z')$ is the solution of Hamilton's equations with $z(t') = z'$. Obviously

$$f_{t,t'}^H = f_{t,0}^H (f_{t',0}^H)^{-1} = f_t^H (f_{t'}^H)^{-1} \quad (25)$$

and the $f_{t,t'}^H$ satisfy the groupoid property

$$f_{t,t'}^H f_{t',t''}^H = f_{t,t''}^H, \quad f_{t,t}^H = I_d \quad (26)$$

for all t, t' and t'' . Notice that it follows in particular that $(f_{t,t'}^H)^{-1} = f_{t',t}^H$.

An essential property which links Hamiltonian dynamics to symplectic geometry is that each mapping f_t^H is a diffeomorphism such that

$$[Df_t^H(z)]^T J Df_t^H(z) = Df_t^H(z) J [Df_t^H(z)]^T = J. \quad (27)$$

Here $Df_t^H(z)$ is the Jacobian matrix of the diffeomorphism f_t^H calculated at the point $z = (x, p)$:

$$Df_t^H(z) = \frac{\partial z_t}{\partial z} = \frac{\partial(x_t, p_t)}{\partial(x, p)}$$

if $z_t = f_t^H(z)$. The equality (27) means that the matrix $Df_t^H(z)$ is symplectic: $Df_t^H(z) \in \text{Sp}(n)$ for every z and t . Any diffeomorphism f of phase space \mathbb{R}^{2n} satisfying the condition

$$Df(z)^T J Df(z) = Df(z) J Df(z)^T = J \quad (28)$$

is called a *symplectomorphism*. Formula (27) thus says that Hamiltonian flows consist of symplectomorphisms, which is a well-known property from classical mechanics [2, 34, 46].

A remarkable fact is that composition and inversion of Hamiltonian flows also yield Hamiltonian flows:

Proposition 14 *Let (f_t^H) and (f_t^K) be the phase flows determined by two Hamiltonian functions $H = H(z, t)$ and $K = K(z, t)$. We have*

$$f_t^H f_t^K = f_t^{H \# K} \quad \text{with} \quad H \# K(z, t) = H(z, t) + K((f_t^H)^{-1}(z), t). \quad (29)$$

$$(f_t^H)^{-1} = f_t^{\bar{H}} \quad \text{with} \quad \bar{H}(z, t) = -H(f_t^H(z), t). \quad (30)$$

Proof. It is based on the transformation properties of the Hamiltonian fields $X_H = J\partial_z H$ under diffeomorphisms; see [22, 34, 46] for detailed proofs. ■

We notice that even if H and K are time-independent Hamiltonians, then $H \# K$ and \bar{H} are time-dependent.

5.2 Hamiltonian isotopies

Formula (28) above shows, using the chain rule, that the symplectomorphisms of \mathbb{R}^{2n} form a group, which we will denote by $\text{Symp}(n)$.

Let us now focus on the Hamiltonian case. We will call a symplectomorphism f such that $f = f_t^H$ for some Hamiltonian function H and time $t = 1$ a *Hamiltonian symplectomorphism*. The choice of time $t = 1$ in this definition is of course arbitrary, and can be replaced with any other choice $t = a$: we have $f = f_{t_a}^{H_a}$ where $H_a(z, t) = aH(z, at)$.

Hamiltonian symplectomorphisms form a subgroup $\text{Ham}(n)$ of the group $\text{Symp}(n)$ of all symplectomorphisms; it is in fact a normal subgroup of $\text{Symp}(n)$ as follows from the formula

$$g^{-1} f_t^H g = f_t^{H \circ g}$$

valid for every symplectomorphism g of \mathbb{R}^{2n} [34, 22, 24]. That $\text{Ham}(n)$ really is a group follows from the two formulas (29) and (30) in Proposition 14 above.

The following result is, in spite of its simplicity, a deep statement about the structure of the group $\text{Ham}(n)$. It says that every continuous path of Hamiltonian transformations passing through the identity is itself the phase flow determined by a certain Hamiltonian function.

Proposition 15 *Let $(f_t)_t$ be a smooth one-parameter family of Hamiltonian transformations such that $f_0 = I_d$. There exists a Hamiltonian function $H = H(z, t)$ such that $f_t = f_t^H$. More precisely, $(f_t)_t$ is the phase flow determined by the Hamiltonian function*

$$H(z, t) = - \int_0^1 z^T J \left(\dot{f}_t \circ f_t^{-1} \right) (\lambda z) d\lambda \quad (31)$$

where $\dot{f}_t = df_t/dt$.

We refer to Wang [49] who gives an elementary proof of formula (31). The result goes back to the seminal paper of Banyaga [4], but the idea is already present in Arnold [2] (p. 269) who uses the apparatus of generating functions.

We will call smooth path $(f_t)_t$ in $\text{Ham}(n)$ joining the identity to some element $f \in \text{Ham}(n)$ a *Hamiltonian isotopy*.

5.3 Symplectic algorithms

Symplectic integrators are designed for the numerical solution of Hamilton's equations; they are algorithms which preserve the symplectic character of Hamiltonian flows. The literature on the topic is immense; a well-cited paper is Channel and Scovel [52]. Among many recent contributions, a highlight is the recent treatise [36] by Kang Feng, Mengzhao Qin; also see the comprehensive paper by Xue-Shen Liu *et al.* [51], and Marsden's online lecture notes [42] (Chapter 9).

Let (f_t^H) be a Hamiltonian flow; we assume first that H is time-independent so that we have the one-parameter group property $f_t^H f_{t'}^H = f_{t+t'}^H$. Choose an initial value z_0 at time $t = 0$. A mapping $f_{\Delta t}$ on \mathbb{R}^{2n} is an algorithm with time step-size Δt for (f_t^H) if we have

$$f_{\Delta t}^H(z) = f_{\Delta t}(z) + O(\Delta t^k);$$

the number k (usually an integer ≥ 1) is called the order of the algorithm. In the theory of Hamiltonian systems one requires that $f_{\Delta t}$ be a symplectomorphism; $f_{\Delta t}$ is then called a symplectic integrator. One of the basic properties one is interested in is convergence: setting $\Delta t = t/N$ (N an integer) when do we have $\lim_{N \rightarrow \infty} (f_{t/N})^N(z) = f_t^H(z)$? One important requirement is stability, *i.e.* $(f_{t/N})^N(z)$ must remain close to z for small t (see Chorin *et al.* [10]).

Here are two elementary examples of symplectic integrators. We assume that the Hamiltonian H has the physical form

$$H(x, p) = U(p) + V(x).$$

- *First order algorithm.* One defines $(x_{k+1}, p_{k+1}) = f_{\Delta t}(x_k, p_k)$ by

$$\begin{aligned} x_{k+1} &= x_k + \partial_p U(p_k - \partial_x V(x_k) \Delta t) \Delta t \\ p_{k+1} &= p_k - \partial_x V(x_k) \Delta t. \end{aligned}$$

- *Second order algorithm.* Setting

$$x'_k = x_k + \frac{1}{2}\partial_p U(p_k)$$

we take

$$\begin{aligned} x_{k+1} &= x'_k + \frac{1}{2}\partial_p U(p_k) \\ p_{k+1} &= p_k - \partial_x V(x'_k)\Delta t. \end{aligned}$$

One can show, using Proposition 15 (Wang [49]), that both schemes are not only symplectic, but also Hamiltonian. For instance, for the first order algorithm above, we have $f_{\Delta t} = f_{\Delta t}^K$ where K is the now time-dependent Hamiltonian

$$K(x, p, t) = U(p) + V(x - \partial_p U(p)t). \quad (32)$$

When the Hamiltonian H is itself time-dependent its flow does no longer enjoy the group property $f_t^H f_{t'}^H = f_{t+t'}^H$, so one has to redefine the notion of algorithm in some way. This can be done by considering the time-dependent flow $(f_{t,t'}^H)$ defined by (25): $f_{t,t'}^H = f_t^H (f_{t'}^H)^{-1}$. One then uses the following trick: define the *suspended flow* (\widetilde{f}_t^H) by the formula

$$\widetilde{f}_t^H(z', t') = (f_{t+t',t'}^H(z'), t + t'); \quad (33)$$

one verifies that the mappings $\widetilde{f}_t^H : \mathbb{R}^{2n} \times \mathbb{R} \longrightarrow \mathbb{R}^{2n} \times \mathbb{R}$ (the “extended phase space”) satisfy the one-parameter group law $\widetilde{f}_t^H \widetilde{f}_{t'}^H = \widetilde{f}_{t+t'}^H$ and one may then define a notion of algorithm approximating \widetilde{f}_t^H (see Struckmeier [48] for the extended phase space approach).

6 Hamiltonian Deformations of Gabor Systems

Let $f \in \text{Ham}(n)$ and $(f_t)_{0 \leq t \leq 1}$ a Hamiltonian isotopy joining the identity to f ; in view of Proposition 15 there exists a Hamiltonian function H such that $f_t = f_t^H$ for $0 \leq t \leq 1$. We want to study the deformation of a \hbar -Gabor frame $\mathcal{G}(\phi, \Lambda)$ by $(f_t)_{0 \leq t \leq 1}$; that is we want to define a deformation

$$\mathcal{G}(\phi, \Lambda) \xrightarrow{f_t} \mathcal{G}(\widehat{U}_t \phi, f_t \Lambda); \quad (34)$$

here \widehat{U}_t is an (unknown) operator associated in some (yet unknown) way with f_t . The fact that when $f_t = S_t \in \text{Sp}(n)$ we have

$$\mathcal{G}(\phi, \Lambda) \xrightarrow{S_t} \mathcal{G}(\widehat{S}_t \phi, S_t \Lambda); \quad (35)$$

where $\widehat{S}_t \in \text{Mp}(n)$ with $S_t = \pi^\hbar(\widehat{S}_t)$ suggests that:

- The operators \widehat{U}_t should be unitary;
- The deformation (34) should reduce to (35) when the isotopy $(f_t)_{0 \leq t \leq 1}$ lies in $\text{Sp}(n)$.

The following property of the metaplectic representation gives us a clue. Let (S_t) be a Hamiltonian isotopy in $\text{Sp}(n) \subset \text{Ham}(n)$. We have seen in Proposition 11 that there exists a Hamiltonian function

$$H(z, t) = \frac{1}{2} M(t) z \cdot z$$

with associated phase flow precisely (S_t) . Consider now the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \widehat{H} \psi \quad , \quad \psi(\cdot, 0) = \psi_0$$

where \widehat{H} is the Weyl quantization of H (it is a formally self-adjoint operator since H is real). It is well-known [22, 24, 19] that $\psi = \widehat{S}_t \psi_0$ where $(\widehat{S}_t)_t$ is the unique path in $\text{Mp}(n)$ passing through the identity and covering (S_t) . This suggests that we should choose $(\widehat{U}_t)_t$ in the following way: let H be the Hamiltonian function determined by the Hamiltonian isotopy (f_t) : $f_t = f_t^H$. Then quantize H into a operator \widehat{H} using the Weyl correspondence, and let (\widehat{U}_t) be the solution of Schrödinger's equation

$$i\hbar \frac{d}{dt} \widehat{U}_t = \widehat{H} \widehat{U}_t \quad , \quad F_0 = I_d.$$

The operators \widehat{U}_t are unitary. Let in fact $u(t) = (\widehat{U}_t \psi | \widehat{U}_t \psi)$ where ψ is in the domain of \widehat{H} (we assume it contains $\mathcal{S}(\mathbb{R}^n)$); we have

$$i\hbar \dot{u}(t) = (\widehat{H} \widehat{U}_t \psi | \widehat{U}_t \psi) - (\widehat{U}_t \psi | \widehat{H} \widehat{U}_t \psi) = 0$$

since \widehat{H} is (formally) self-adjoint; it follows that $(\widehat{U}_t \psi | \widehat{U}_t \psi) = (\psi | \psi)$.

While definition (35) of a Hamiltonian deformation of a Gabor system is “reasonable”, its practical implementation is difficult because it requires the solution of a Schrödinger equation. We will try to find a weaker, more tractable definition of the correspondence (34), which is easier to implement numerically.

6.1 The semiclassical approach

The idea of this method comes from semiclassical mechanics; historically it seems to be due to Heller [33]. Hagedorn [30, 31] has given the method

a firm mathematical basis; also see Littlejohn's review paper [39] and the construction in [44].

For fixed z_0 we set $z_t = f_t^H(z_0)$ and define the new Hamilton function

$$H_{z_0}(z, t) = (\partial_z H)(z_t, t)(z - z_t) + \frac{1}{2} D_z^2 H(z_t, t)(z - z_t)^2; \quad (36)$$

it is the Taylor series of H at z_t with terms of order 0 and > 2 suppressed. The corresponding Hamilton equations are

$$\dot{z} = J \partial_z H(z_t, t) + J D_z^2 H(z_t, t)(z - z_t). \quad (37)$$

We make the following obvious but essential observation: in view of the uniqueness theorem for the solutions of Hamilton's equations, the solution of (37) with initial value z_0 is the same as that of the Hamiltonian system

$$\dot{z}(t) = J \partial_z H(z(t), t) \quad (38)$$

with $z(0) = z_0$. Denoting by $(f_t^{H_{z_0}})$ the Hamiltonian flow determined by H_{z_0} we thus have $f_t^H(z_0) = f_t^{H_{z_0}}(z_0)$. More generally, the flows $(f_t^{H_{z_0}})$ and (f_t^H) are related by a simple formula:

Proposition 16 *The solutions of Hamilton's equations (37) and (38) are related by the formula*

$$z(t) = z_t + S_t(z(0) - z_0) \quad (39)$$

where $z_t = f_t^H(z_0)$, $z(t) = f_t^H(z)$ and $(S_t)_t$ is the phase flow determined by the quadratic time-dependent Hamiltonian

$$H^0(z, t) = \frac{1}{2} D_z^2 H(z_t, t) z \cdot z. \quad (40)$$

Equivalently,

$$f_t^H(z) = T[z_t - S_t(z_0)] S_t(z(0)) \quad (41)$$

where $T(\cdot)$ is the translation operator.

Proof. Let us set $u = z - z_t$. We have, taking (37) into account,

$$\dot{u} + \dot{z}_t = J \partial_z H(z(t), t) + J D_z^2 H(z_t, t) u$$

that is, since $\dot{z}_t = J \partial_z H(z_t, t)$,

$$\dot{u} = J D_z^2 H(z_t, t) u.$$

It follows that $u(t) = S_t(u(0))$ and hence

$$z(t) = f_t^H(z_0) + S_t u(0) = z_t - S_t(z_0) + S_t(z(0))$$

which is precisely (39). ■

The nearby orbit method (at order $N = 0$) consists in making the Ansatz that the approximate solution to Schrödinger's equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi \quad , \quad \psi(\cdot, 0) = \phi_{z_0}^\hbar$$

where

$$\phi_{z_0}^\hbar = \hat{T}^\hbar(z_0)\phi_0^\hbar \tag{42}$$

is the standard coherent state centered at z_0 is given by the formula

$$\tilde{\psi}(x, t) = e^{\frac{i}{\hbar}\gamma(t, z_0)} \hat{T}^\hbar(z_t) \hat{S}_t(z_0) \hat{T}^\hbar(z_0)^{-1} \phi_{z_0}^\hbar \tag{43}$$

where the phase $\gamma(t, z_0)$ is the symmetrized action

$$\gamma(t, z_0) = \int_0^t \left(\frac{1}{2} \sigma(z_{t'}, \dot{z}_{t'}) - H(z_{t'}, t') \right) dt' \tag{44}$$

calculated along the Hamiltonian trajectory leading from z_0 at time $t_0 = 0$ to z_t at time t . One shows that under suitable conditions on the Hamiltonian H the approximate solution satisfies, for $|t| \leq T$, an estimate of the type

$$\|\psi(\cdot, t) - \tilde{\psi}(\cdot, t)\| \leq C(z_0, T) \sqrt{\hbar} |t| \tag{45}$$

where $C(z_0, T)$ is a positive constant depending only on the initial point z_0 and the time interval $[-T, T]$ (Hagedorn [30, 31]).

Remark 17 *Formula (43) shows that the solution of Schrödinger's equation with initial datum ϕ_0^\hbar is approximately the Gaussian obtained by propagating ϕ_0^\hbar along the Hamiltonian trajectory starting from $z = 0$ while deforming it using the metaplectic lift of the linearized flow around this point.*

6.2 Application to Gabor frames

Let us state and prove the main results of this paper.

In what follows we consider a Gaussian Gabor system $\mathcal{G}(\phi_0^\hbar, \Lambda)$; applying the nearby orbit method to ϕ_0^\hbar yields the approximation

$$\phi_t^\hbar = e^{\frac{i}{\hbar}\gamma(t, 0)} \hat{T}^\hbar(z_t) \hat{S}_t \phi_0^\hbar \tag{46}$$

where we have set $\hat{S}_t = \hat{S}_t(0)$. Let us consider the Gabor system $\mathcal{G}(\phi_t^\hbar, \Lambda_t)$ where $\Lambda_t = f_t^H(\Lambda)$.

Proposition 18 *The Gabor system $\mathcal{G}(\phi_t^h, \Lambda_t)$ is a Gabor \hbar -frame if and only if $\mathcal{G}(\phi_0^h, \Lambda)$ is a Gabor \hbar -frame; when this is the case both frames have the same bounds.*

Proof. Writing

$$I_t(\psi) = \sum_{z \in \Lambda_t} |(\psi | \widehat{T}^h(z) \phi_t^h)|^2 \quad (47)$$

we set out to show that the inequality

$$a \|\psi\|^2 \leq I_t(\psi) \leq b \|\psi\|^2 \quad (48)$$

(for all $\psi \in L^2(\mathbb{R}^n)$) holds for every t if and only if it holds for $t = 0$ (for all $\psi \in L^2(\mathbb{R}^n)$). In view of definition (46) we have

$$I_t(\psi) = \sum_{z \in \Lambda_t} |(\psi | \widehat{T}^h(z) \widehat{T}^h(z_t) \widehat{S}_t \phi_0^h)|^2;$$

the commutation formula (18) yields

$$\widehat{T}^h(z) \widehat{T}^h(z_t) = e^{i\sigma(z, z_t)/\hbar} \widehat{T}^h(z_t) \widehat{T}^h(z)$$

and hence

$$\begin{aligned} I_t(\psi) &= \sum_{z \in \Lambda_t} |(\psi | \widehat{T}^h(z_t)) \widehat{T}^h(z) \widehat{S}_t \phi_0^h|^2 \\ &= \sum_{z \in \Lambda} |(\psi | \widehat{T}^h(z_t) \widehat{T}^h(f_t^H(z)) \widehat{S}_t \phi_0^h)|^2. \end{aligned}$$

Since $\widehat{T}^h(z_t)$ is unitary the inequality (48) is thus equivalent to

$$a \|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi | \widehat{T}^h(f_t^H(z)) \widehat{S}_t \phi_0^h)|^2 \leq b \|\psi\|^2. \quad (49)$$

In view of formula (39) we have, since $S_t z_0 = 0$ because $z_0 = 0$,

$$f_t^H(z) = S_t z + f_t^H(0) = S_t z + z_t$$

hence the inequality (49) can be written

$$a \|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi | \widehat{T}^h(S_t z + z_t) \widehat{S}_t \phi_0^h)|^2 \leq b \|\psi\|^2. \quad (50)$$

In view of the product formula (19) for Heisenberg–Weyl operators we have

$$\widehat{T}^h(S_t z + z_t) = e^{i\sigma(S_t z, z_t)/2\hbar} \widehat{T}^h(z_t) \widehat{T}^h(S_t z)$$

so that (50) becomes

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\hat{T}^h(z_t)\hat{T}^h(S_t z)\hat{S}_t\phi_0^h)|^2 \leq b\|\psi\|^2; \quad (51)$$

the unitarity of $\hat{T}^h(z_t)$ implies that (51) is equivalent to

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\hat{T}^h(S_t z)\hat{S}_t\phi_0^h)|^2 \leq b\|\psi\|^2. \quad (52)$$

Using the symplectic covariance formula (6) we have

$$\hat{T}^h(S_t z)\hat{S}_t = \hat{S}_t\hat{T}^h(z)$$

so that the inequality (52) can be written

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\hat{S}_t\hat{T}^h(z)\phi_0^h)|^2 \leq b\|\psi\|^2;$$

since \hat{S}_t is unitary, this is equivalent to

$$a\|\psi\|^2 \leq \sum_{z \in \Lambda} |(\psi|\hat{T}^h(z)\phi_0^h)|^2 \leq b\|\psi\|^2.$$

The Proposition follows. ■

The fact that we assumed that the window is the centered coherent state ϕ_0^h is not essential. For instance, Proposition 13 shows that the result remains valid if we replace ϕ_0^h with a coherent state having arbitrary center, for instance $\phi_{z_0}^h = \hat{T}^h(z_0)\phi_0^h$. More generally:

Corollary 19 *Let $\mathcal{G}(\phi, \Lambda)$ be a Gabor system where the window ϕ is the Gaussian*

$$\phi_M^h(x) = \left(\frac{\det \operatorname{Im} M}{(\pi \hbar)^n} \right)^{1/4} e^{\frac{i}{2\hbar} M x \cdot x} \quad (53)$$

where $M = M^T$, $\operatorname{Im} M > 0$. Then $\mathcal{G}(\phi_t^h, \Lambda_t)$ is a Gabor \hbar -frame if and only if it is the case for $\mathcal{G}(\phi, \Lambda)$.

Proof. It follows from the properties of the action of the metaplectic group on Gaussians (see Appendix A) that there exists $\hat{S} \in \operatorname{Mp}(n)$ such that $\phi_M^h = \hat{S}\phi_0^h$. Let $S = \pi^h(\hat{S})$ be the projection on $\operatorname{Sp}(n)$ of \hat{S} ; the Gabor system $\mathcal{G}(\phi_M^h, \Lambda)$ is a \hbar -frame if and only if $\mathcal{G}(\hat{S}^{-1}\phi_M^h, S^{-1}\Lambda) = \mathcal{G}(\phi_0^h, S^{-1}\Lambda)$ is a \hbar -frame in view of Proposition 5. The result now follows from Proposition 18. ■

7 Discussion and Additional Remarks

We have given one working definition of the notion of Hamiltonian deformation of a Gabor frame; this definition uses ideas from semiclassical mechanics. However, we have used no approximations. We could therefore call this deformation scheme “weak Hamiltonian deformation”. An important remark is that in all our results, one can assume that the window ϕ belongs to the Feichtinger algebra $S_0(\mathbb{R}^n)$ (reviewed in Appendix B). This is due to the fact that we have transformed the Gabor frames under consideration only by the phase space translations $\widehat{T}^h(z)$ and by metaplectic operators; it turns out that $S_0(\mathbb{R}^n)$ is the smallest Banach algebra invariant under these operations, and thus semiclassical propagation preserve the Feichtinger algebra (see de Gosson [23]). A consequence is that the weak Hamiltonian deformation scheme behaves well with respect to the Feichtinger algebra. It is unknown whether this property is conserved under passage to the general definition (35), that is

$$\mathcal{G}(\phi, \Lambda) \xrightarrow{f_t} \mathcal{G}(\widehat{U}_t \phi, f_t \Lambda) \quad (54)$$

where \widehat{U}_t is the solution of the Schrödinger equation associated with the Hamiltonian operator corresponding to the Hamiltonian isotopy $(f_t)_{0 \leq t \leq 1}$. This because one does not know at the time of writing if the solution to Schrödinger equations with initial data in $S_0(\mathbb{R}^n)$ also is in $S_0(\mathbb{R}^n)$ for given time t .

Since our definition of weak deformations was motivated by semiclassical considerations one could perhaps consider refinements of this method using the asymptotic expansions of Hagedorn [30, 31] and his followers; this could then lead to “higher order” weak deformations, depending on the number of terms that are retained. Still, there remains the question of the general definition (54) where the exact quantum propagator is used. It would indeed be more intellectually (and also probably practically!) satisfying to study this definition in detail. As we said, we preferred in this first approach to consider a weaker version because it is relatively easy to implement numerically using symplectic integrators. The general case (54) is challenging, but not probably out of reach. From a theoretical point of view, it amounts to construct an extension of the metaplectic representation in the non-linear case; that such a representation indeed exists has been shown in our paper with Hiley [25] (a caveat: one sometimes finds in the physical literature a claim following which such an extension could not be constructed; a famous theorem of Groenewold and Van Hove being invoked. This is merely a misunderstanding of this theorem, which only claims that there is no way to

extend the metaplectic representation so that the Dirac correspondence between Poisson brackets and commutators is preserved). There remains the problem of how one could prove that the deformation scheme (54) preserves the frame property; a possible approach could consist in using a time-slicing (as one does for symplectic integrators); this would possibly also lead to some insight on whether the Feichtinger algebra is preserved by general quantum evolution.

APPENDIX A: METAPLECTIC GROUP AND GAUSSIANS

Let $\text{Mp}(n)$ be the metaplectic representation of the symplectic group $\text{Sp}(n)$ (see [19, 22, 24]); it is a unitary representation of the double cover $\text{Sp}_2(n)$ of $\text{Sp}(n)$: we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Mp}(n) \xrightarrow{\pi^h} \text{Sp}(n) \longrightarrow 0$$

where $\pi^h : \widehat{S} \mapsto S$ is the covering projection; we explain the appearance of the subscript h below. The metaplectic group is generated by the following elementary operators:

- *Fourier transform*:

$$\widehat{J}\psi = \left(\frac{1}{2\pi i h}\right)^{n/2} \int_{\mathbb{R}^n} e^{-i x x' / h} \psi(x') dx' \quad (55)$$

(notice the presence of the imaginary unit i in the prefactor);

- *Unitary dilations*:

$$\widehat{M}_{L,m}\psi = i^m \sqrt{|\det L|} \psi(Lx) \quad (\det L \neq 0) \quad (56)$$

where m is an integer depending on the sign of $\det L$: $m \in \{0, 2\}$ if $\det L > 0$ and $m \in \{1, 3\}$ if $\det L < 0$;

- “*Chirps*”:

$$\widehat{V}_P = e^{-i P x^2 / 2h} \psi(x) \quad (P = P^T). \quad (57)$$

The projections on $\text{Sp}(n)$ of these operators are given by $\pi^h(\widehat{J}) = J$, $\pi^h(\widehat{M}_{L,m}) = M_L$, and $\pi^h(\widehat{V}_P) = V_P$ with

$$M_L = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}, \quad V_P = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \quad (58)$$

(the matrices V_P are sometimes called “symplectic shears”).

The projection of a covering group onto its base group is defined only up to conjugation; our choice –and notation– is here dictated by the fact that to the \hbar -dependent operators (55) and (56) should correspond the symplectic matrices (58). For instance, in time-frequency analysis it is customary to make the choice $\hbar = 1/2\pi$. Following formula relates the projections π^{\hbar} and $\pi = \pi^{1/2\pi}$:

$$\pi(\widehat{S}) = \pi^{\hbar}(\widehat{M}_{1/\sqrt{2\pi\hbar}}\widehat{S}\widehat{M}_{\sqrt{2\pi\hbar}}) \quad (59)$$

where $\widehat{M}_{\sqrt{2\pi\hbar}} = \widehat{M}_{\sqrt{2\pi\hbar}I,0}$.

Metaplectic operators are not only unitary operators on $L^2(\mathbb{R}^n)$ but also linear automorphisms of $\mathcal{S}(\mathbb{R}^n)$ which extend by duality to automorphisms of $\mathcal{S}'(\mathbb{R}^n)$.

There is an alternative way to describe the metaplectic group $\text{Mp}(n)$. Let

$$W(x, x') = \frac{1}{2}Px^2 - Lx \cdot x' + \frac{1}{2}Qx'^2 \quad (60)$$

where P, L, Q are real $n \times n$ matrices, P and Q symmetric and L invertible (we are writing Px^2 for $Px \cdot x$, etc.). Let m be a choice of $\arg \det L$ as in formula (56); each $\widehat{S} \in \text{Mp}(n)$ is the product to two operators of the type

$$\widehat{S}_{W,m}\psi(x) = \left(\frac{1}{2\pi i\hbar}\right)^{n/2} i^m \sqrt{|\det L|} \int_{\mathbb{R}^n} e^{-iW(x,x')/\hbar} \psi(x') dx' \quad (A1)$$

(see [38, 22, 24]). The operators $\widehat{S}_{W,m}$ can be factorized as

$$\widehat{S}_{W,m} = \widehat{V}_{-P} \widehat{M}_{L,m} \widehat{J} \psi \widehat{V}_{-Q} \quad (A2)$$

and hence belong to $\text{Mp}(n)$. The projection $S_W = \pi^{\hbar}(\widehat{S}_{W,m})$ is characterized by the condition

$$(x, p) = S_W(x', p') \iff \begin{cases} p = \partial_x W(x, x') \\ p' = -\partial_x W(x, x') \end{cases} ;$$

this condition identifies W with the generating function of first type, familiar from Hamiltonian mechanics [2, 22, 24, 38]. A straightforward calculation using the expression (60) of W yields the symplectic matrix

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}.$$

The metaplectic group acts on Gaussian functions in a particularly simple way. Let M be a complex $n \times n$ matrix; we assume in fact that M belongs to the Siegel half-space

$$\Sigma_n^+ = \{M : M = M^T, \text{Im } M > 0\}.$$

We call generalized centered coherent state a Gaussian function of the type

$$\phi_M^h(x) = \left(\frac{\det \operatorname{Im} M}{(\pi h)^n} \right)^{1/4} e^{\frac{i}{2h} M x \cdot x} \quad (61)$$

and for $z_0 \in \mathbb{R}^{2n}$ we set

$$\phi_{M,z_0}^h = \widehat{T}^h(z_0) \phi_M^h \quad (62)$$

(it is a Gaussian centered at the point z_0). The symplectic group $\operatorname{Sp}(n)$ acts transitively on the Siegel half-space via the law [19]

$$(S, M) \mapsto \alpha(S)M = (C + DM)(A + BM)^{-1}. \quad (63)$$

One can show [24] that if $M = X + iY$ then

$$X = -(CA^T + DB^T)(AA^T + BB^T)^{-1} \quad (64)$$

$$Y = (AA^T + BB^T)^{-1}. \quad (65)$$

This action induces in turn a transitive action

$$(\widehat{S}, \phi_{M,z_0}^h) \mapsto \phi_{\alpha(S)M, Sz_0}^h$$

of the metaplectic group $\operatorname{Mp}(n)$ on the set G_n of Gaussians of the type (62). These actions make the following diagram

$$\begin{array}{ccc} \operatorname{Mp}(n) \times G_n & \longrightarrow & G_n \\ \downarrow & & \downarrow \\ \operatorname{Sp}(n) \times \Sigma_n^+ & \longrightarrow & \Sigma_n^+ \end{array}$$

commutative (the vertical arrows being the mappings $(\widehat{S}, \phi_{M,z_0}^h) \mapsto (S, M)$ and $\phi_{M,z_0}^h \mapsto M$, respectively).

The formulas above can be proven by using either the properties of the Wigner transform, or by a calculation of Gaussian integrals using the operators $\widehat{S}_{W,m}$ defined by formula (A1).

APPENDIX B: FEICHTINGER'S ALGEBRA

The Feichtinger algebra $S_0(\mathbb{R}^n)$ was introduced in [13, 14, 15]; it is an important particular case of the modulation spaces defined by the same author; we refer to Gröchenig's treatise [27] for a complete study of these important functional spaces. Also see Feichtinger and Luef [18] for an up to date concise review.

The Feichtinger algebra is usually defined in terms of short-time Fourier transform

$$V_\phi\psi(z) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x'} \psi(x') \overline{\phi(x' - x)} dx'; \quad (66)$$

which is related to the cross-Wigner transform by the formula

$$W(\psi, \phi)(z) = \left(\frac{2}{\pi h}\right)^{n/2} e^{\frac{2i}{h} p \cdot x} V_{\phi_{\sqrt{2\pi h}}} \psi_{\sqrt{2\pi h}}(z \sqrt{\frac{2}{\pi h}}) \quad (67)$$

where $\psi_{\sqrt{2\pi h}}(x) = \psi(x\sqrt{2\pi h})$ and $\phi^\vee(x) = \phi(-x)$; equivalently

$$V_\phi\psi(z) = \left(\frac{2}{\pi h}\right)^{-n/2} e^{-i\pi p \cdot x} W(\psi_{1/\sqrt{2\pi h}}, \phi_{1/\sqrt{2\pi h}}^\vee)(z \sqrt{\frac{\pi h}{2}}). \quad (68)$$

The Feichtinger algebra $S_0(\mathbb{R}^n)$ consists of all $\psi \in \mathcal{S}'(\mathbb{R}^n)$ such that $V_\phi\psi \in L^1(\mathbb{R}^{2n})$ for every window ϕ . In view of the relations (67), (68) this condition is equivalent to $W(\psi, \phi) \in L^1(\mathbb{R}^{2n})$. A function $\psi \in L^2(\mathbb{R}^n)$ belongs to $S_0(\mathbb{R}^n)$ if and only if $W\psi \in L^1(\mathbb{R}^{2n})$; here $W\psi = W(\psi, \psi)$ is the usual Wigner function. The number

$$\|\psi\|_{\phi, S_0}^h = \|W(\psi, \phi)\|_{L^1(\mathbb{R}^{2n})} = \int_{\mathbb{R}^{2n}} |W(\psi, \phi)(z)| dz \quad (69)$$

is the norm of ψ relative to the window ϕ . We have the inclusions

$$\mathcal{S}(\mathbb{R}^n) \subset S_0(\mathbb{R}^n) \subset C^0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \cap F(L^1(\mathbb{R}^n)) \quad (70)$$

where $F(L^1(\mathbb{R}^n))$ is the image of $L^1(\mathbb{R}^n)$ by the Fourier transform. One proves that $S_0(\mathbb{R}^n)$ is an algebra, both for pointwise multiplication and for convolution.

An essential property of the Feichtinger algebra is that it is closed under the action of the Weyl-metaplectic group $\text{WMp}(n)$: if $\psi \in S_0(\mathbb{R}^n)$, $\hat{S} \in \text{Mp}(n)$, and $z_0 \in \mathbb{R}^n$ we have both $\hat{S}\psi \in S_0(\mathbb{R}^n)$ and $\hat{T}^h(z_0)\psi \in S_0(\mathbb{R}^n)$. In particular $\psi \in S_0(\mathbb{R}^n)$ if and only if $F\psi \in S_0(\mathbb{R}^n)$. One proves that $S_0(\mathbb{R}^n)$ is the smallest Banach space containing $\mathcal{S}(\mathbb{R}^n)$ and having this property.

Acknowledgement 20 *This work has been supported by a research grant from the Austrian Research Agency FWF (Projekt Nummer P23902-N13).*

Acknowledgement 21 *The author wishes to thank Hans Feichtinger and Karlheinz Gröchenig for valuable suggestions, criticism, and discussions about Gabor frames.*

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